Shocks to Capital Intensity Make the Solow Equation an Impulsive Differential Equation

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1. Economic shocks

- sources: nature, market, government
- nature: Kobe earthquake of 17.01.1995, 5,273 people died, 95 and 250 billion U.S rebuilding costs [Sommerville (1995)]
- market: oil crises of October 1973: the OPEC announce that they would cut crude oil production and increase prices (Arab-Israeli war)
- government: nationalisation, privatisation, deregulation

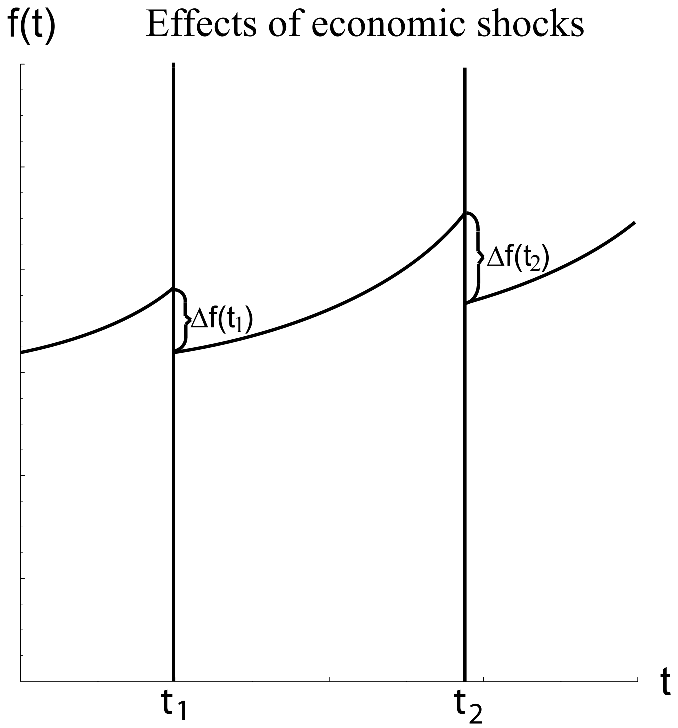
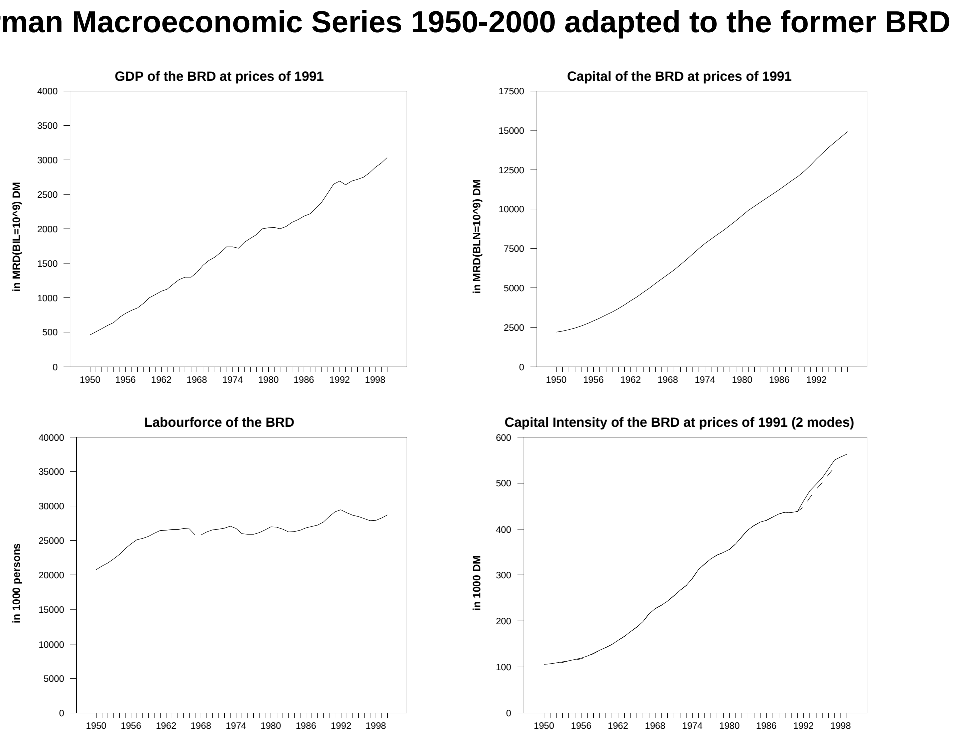


Figure 1: Appearance of economic shocks in macroeconomic variables

2. The analysed time series



- the *capital stock* $K(t)$, the labor force $L = L(t)$, the capital intensity: $r(t) = K(t)/L(t)$ and $GDP(t)$

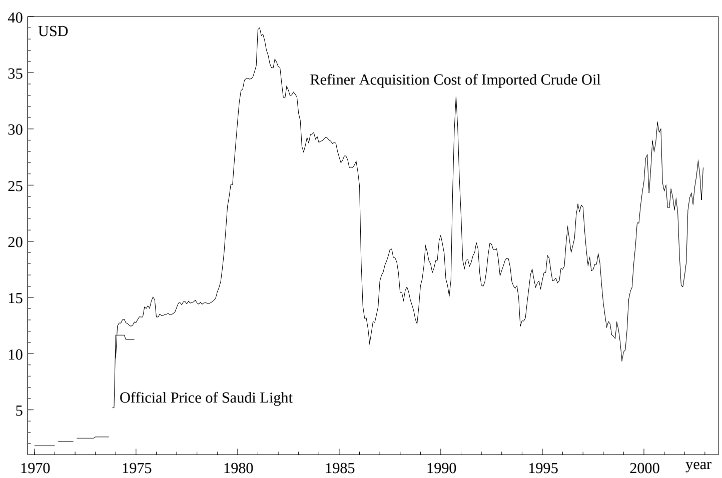


Figure 2: World Oil Price Chronology: 1970-2002 (Source: Energy Information Administration, Washington, USA

3. The Solow equation

- Solow's neoclassical model (1956), with the *marginal propensity to save* s, the *population growth rate* n, leads to his seminal equation

$$\dot{K}(t) = sF(K(t), L_0 e^{nt}) \tag{1}$$

- based on the strict Cobb-Douglas production function with the the *production-elasticity of capital* α, $1 > \alpha > 0$

$$Y = F(K, L) = AK^\alpha L^{1-\alpha} \tag{2}$$

- the solution is a *convergent* process for the capital intensity $r(t) = K(t)/L(t)$

$$r(t) = [(r(0)^{1-\alpha} - \frac{s}{n})e^{-n(1-\alpha)t} + \frac{s}{n}]^{\frac{1}{1-\alpha}} \xrightarrow[t\to\infty]{} (\frac{s}{n})^{\frac{1}{1-\alpha}} \tag{3}$$

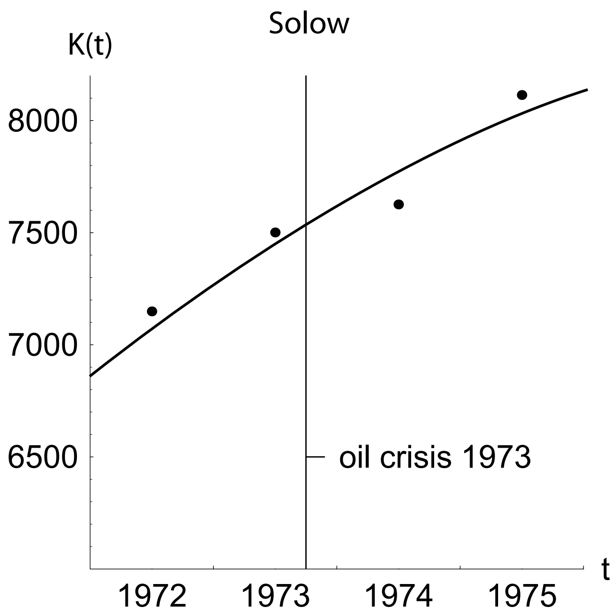


Figure 3: The continuous capital function $K(t)$ of Solow, superposed to real German capital data (in Billion of DM)

4. Does the oil shock of 1973 have a permanent impact on GNP?

- The Keynesians, monetarists and new classical economists alike until the early 1980 accepted Solow's neoclassical model that the long-run trend component of GNP is smooth, with short-run fluctuations about trend being primarily determined by demand shocks.
- Based on empirical research into the US data, Nelson and Plosser (1982) have been *unable to reject the hypothesis* that macroeconomic time series, like GNP or employment, follow a *random walk*.
- The result of Nelson and Plosser (1982) means that a one-unit-shock to GNP affects forecast forever by one unit. But this result does not appear to be evident.

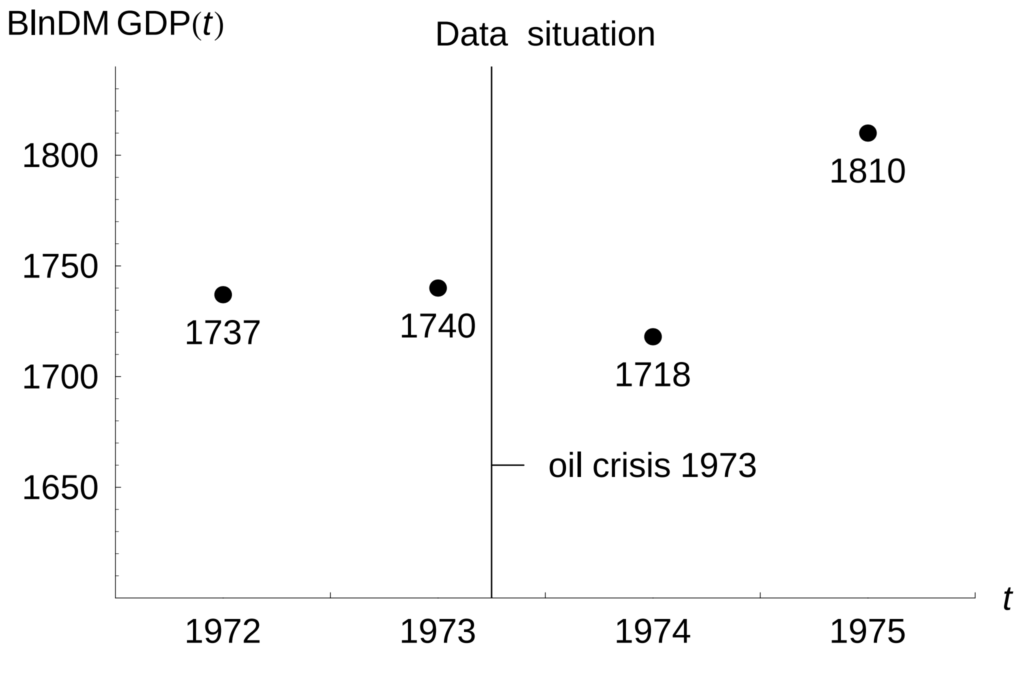


Figure 4: Data of German GDP (Bundesamt für Statistik

5. How big is the random walk in macroeconomic time series?

- Thus, Cochrane (1988) re-examined the long-run properties of GNP.
- Commonly, there are two ways of modelisation
- a) *trend stationary processes* (TS)

$$\begin{aligned} y_t &= \alpha + \beta t + z_t \\ \phi(L) z_t &= \theta(L) u_t \ ; \ u_t \sim iid(0, \sigma_u^2) \end{aligned} \tag{4}$$

- Cochrane (1988) formulates the TS models (4), using the property of invertibility of z_t

$$y_t = \alpha + \beta t + \sum_{j=0}^{\infty} a_j \cdot \varepsilon_{t-j} \ , \tag{5}$$

- A simple form of a TS process (5) for $\beta = 0$ is an $AR(1)$ process with drift α'

$$y_t = \alpha' + \gamma y_{t-1} + \varepsilon_t \quad 0 < \gamma < 1, \tag{6}$$

- b) The counter-part of the linear TS process is the first-order *difference stationary processes* (DS) (7) of Nelson and Plosser (1982).

$$\begin{aligned} y_t &= \alpha + y_{t-1} + z_t, \\ \phi(L) z_t &= \theta(L) u_t \ ; \ u_t \sim iid(0, \sigma_u^2) \end{aligned} \tag{7}$$

- The simplest example of (7) is the *random walk*, the z_t are serially uncorrelated

6. Unit roots and the variance ratio statistic

- The difference between the TS models and DS models is dramatic. In the TS approach, the impact of the shock generally dies out, since in the DS approach, modelled by (7), the effect is *permanent*.

- When the macroeconomic time series are described by DS models, the presence of unit roots has to be evaluated.

- Cochrane (1988) proposes for the *measurement of persistence* to evaluate the presence of a unit root with the *variance ratio (VR) statistic* (VR_p).

$$\widehat{VR}_p = \frac{var(y_t - y_{t-p})}{p \cdot var(y_t - y_{t-1})} \cdot (\frac{T}{T-p+1}), \tag{8}$$

- It is well-known that if $\{y_t\}$ is a pure random walk, then $VR_p \to 1$ for $p \to \infty$. If $\{y_t\}$ is trend-stationary, then $VR_p \to 0$ for $p \to \infty$.

- An $ARMA(p, 1, q)$ process can be decomposed (Beveridge and Nelson (1981))

$$y_t = x_t + c_t, \tag{9}$$

where $\{x_t\}$ is the random walk component and $\{c_t\}$ is the stationary component.

- Cochrane (1988) suggests that the variance of the random walk component is about *one third* of the the variance of the year-to-year changes of US GDP. Thus, the world is in harmony between *continuous* and *discrete* processes.

7. The four German macroeconomic time series

- The annual German macroeconomic time series of $GDP(t)$, capital stock $K(t)$, capital intensity $r(t) = K(t)/L(t)$, all at prices of 1991 and the labour force $L(t)$, $1950 - 2000$.

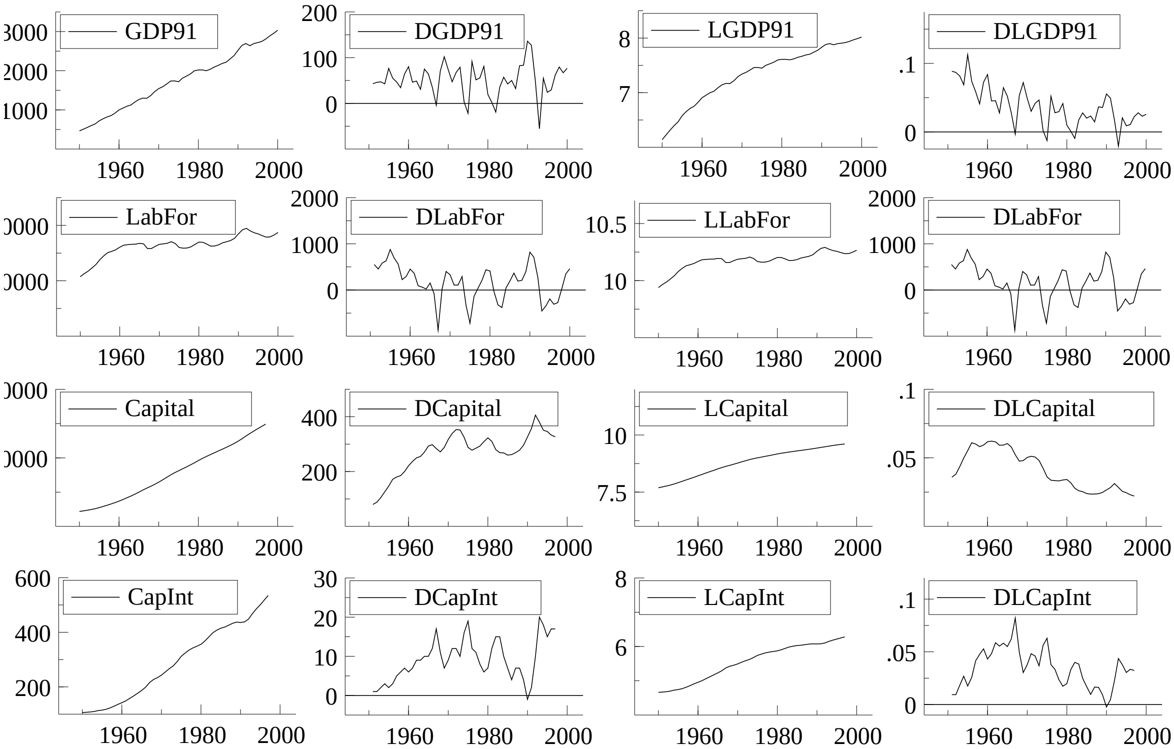


Figure 5: German Macroeconomic Time Series (PcGive)

- The presence of random walk in the German series!
- Clearly, from the ADF tests none of the German time series GDP, L, K, r exhibits stationarity, the four series appear to be at least $I(1)$.
- Thus, the method of Cochrane (1988) with the variance ratio statistic (8) has been applied to the present German series.

The variance ratio test of Cochrane

- Variance ratio values have been calculated. The results are presented with the sixteen plots of Fig. 7. Obviously none of them shows convergence to 0.
- This means, none of them shows pure stationarity.

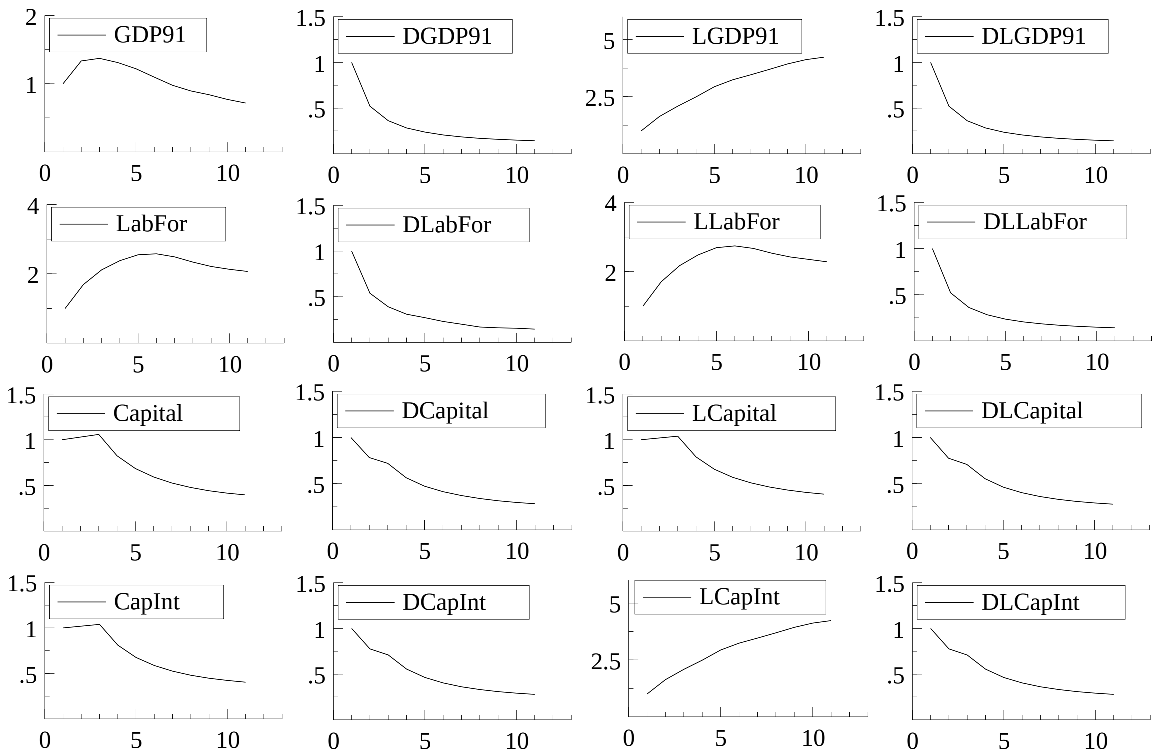


Figure 6: Variance ratio statistics of the German Macroeconomic Series, (PcGive)

- The persistence of the random walk component is explored with the variance ratio statistic. The hypothesis of the presence of a *random walk component* and therefore also a unit root in each of the German series GDP, L, K, r and $log(GDP)$, $log(L)$ cannot be refused.
- Therefore, the German series GDP, L, K, r and the series $log(GDP)$, $log(L)$ appear to be at least integrated $I(1)$.

The Capital is I(2) - the Capital intensity is I(1)

- The German annual time series of *capital intensity* $r(t)$ appears to be I(1).
- The ADF-test results show that the null hypothesis of a unit root is not rejected in the time series $K(t)$ and $\Delta K(t)$
- The German annual time series of *capital intensity* $r = K/L$ is modelled as an I(1) time series, the Beveridge (1981) decomposition scheme becoming applicable, $r(t)$ is decomposed into a *random walk* component and a *stationary* component (9).
- The German annual time series of *capital* $K(t)$ appears to be I(2).

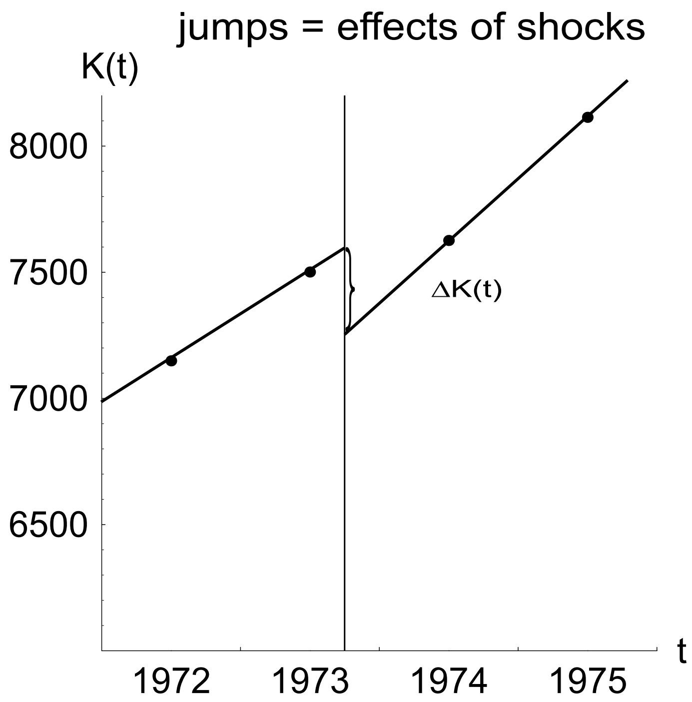


Figure 7: The German capital (in Billions of DM) exhibits jumps (Bundesamt für Statistik)

8. The Impulsive Solow equation

- Model the jumps of German capital $K(t)$ and the capital intensity $r(t) = \frac{K(t)}{L(t)}$ with *real-valued piece-wise continuous functions.*
- Let $\tau_1, \tau_2, ..., \tau_k > 0$ be the moments, when the stock of capital $K(t)$ is subject to shock effects changing from the positions $K(\tau_k - 0)$ into the position $K(\tau_k + 0)$.

$$\begin{aligned} \dot{K}(t) &= sF(K(t), L_0 e^{nt}),\ t > t_0,\ t \neq \tau_k, \\ K(t_0 + 0) &= K_0 \\ \Delta K(t) &= J_k(K(t)),\ t = \tau_k,\ k = 1, 2, ..., \end{aligned} \tag{10}$$

- where functions J_k characterize the magnitude of the impulse effect at the times τ_k; $K(\tau_k - 0)$ and $K(\tau_k + 0)$ are respectively the capital level before and after the impulsive effect; K_0 is the initial capital.
- The solution of the impulsive Solow equation

$$\begin{aligned} K(t) = L_0 e^{nt}[(r(t_0)^{1-\alpha} - \frac{s}{n})e^{-n(1-\alpha)t} + \frac{s}{n}]^{\frac{1}{1-\alpha}} + \\ \sum_{t_0 \leq \tau_k < t} J_k(K(\tau_k)) \end{aligned} \tag{11}$$

$$r(t) = [(r(t_0)^{1-\alpha} - \frac{s}{n})e^{-n(1-\alpha)t} + \frac{s}{n}]^{\frac{1}{1-\alpha}} + \sum_{t_0 \leq \tau_k < t} B_k(r(\tau_k)) \tag{12}$$

- where the shock effects are $B_k(r(\tau_k)) = r(\tau_k + 0) - r(\tau_k), k = 1, 2, ...$

9. Conclusions

- The variance ratio statistic of Cochrane (1988) and the augmented Dickey-Fuller (ADF) test are applied to analyse the *unit roots* in four German macro-economic time series: the gross domestic product (GDP), the capital, the labour force and the capital intensity.

- The data support the hypothesis that the German capital intensity is integrated I(1), exhibiting the presence of a *random walk component* and the German capital is integrated I(2).

- The seminal differential equation of Solow (1956) is revisited and becomes an impulsive differential Solow equation, when shocks to capital intensity are considered.

- The result of the present analysis: There exist *real valued functions* of the German *capital intensity* and the German *capital* modelled with jumps, solved for the presupposed strict Cobb-Douglas production function.

- Simple data problem: An estimation of the overall German parameters α, s and n has to be performed.

- Open data problem: The measurement and establishment of the jumps $B_k(r(\tau_k))$ of the capital intensity $r(t)$, respectively the jumps $J_k(K(\tau_k))$ of the capital function $K(t)$ at some time points $\tau_k, k =$1973:10,... has to be performed.