Shocks to Capital Intensity Make the Solow Equation an Impulsive Differential Equation

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Abstract

The seminal differential equation of Solow (1956) becomes an impulsive differential equation, when shocks to capital intensity are modelled with jumps. This statement results from an analysis of unit roots in four German macroeconomic time series: the gross domestic product (GDP), the capital, the labour force and the capital intensity. The variance ratio statistic of Cochrane (1988) and the augmented Dickey-Fuller (ADF) test are applied to analyse the unit roots. The data support the hypothesis that the German capital intensity is integrated I(1), exhibiting the presence of a *random walk component* in application to the Beveridge decomposition (1981) scheme. Consequently, the real value function of capital intensity is modelled with jumps and makes the Solow equation an impulsive Solow equation that is presently solved for presupposed strict Cobb-Douglas production functions.

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1. Introduction

Based on empirical research into the US data, Nelson and Plosser (1982) have been *unable to reject the hypothesis* that macroeconomic time series, like GNP or employment, follow a *random walk*. This means that a one-unit-shock to GNP affects forecast forever by one unit. But this result does not appear to be evident. Thus, for Cochrane (1988) the world is in harmony between *continuous* and *discrete* processes. Cochrane reexamined the long-run properties of GNP and found that GNP can, in fact revert toward a "trend" following a shock. These macroeconomic time series are typically modelled by AR(1) processes, (see Snowdon, 1994, p. 241), any shock modelled on a random disturbance variable. Under the assumption that the GNP of the US is a first-difference stationary process, its fluctuations are viewed partly as temporary, partly as permanent deviations from "potential GNP", due to the decomposition technique of Beveridge and Nelson (1981). Cochrane proposes to use the *variance ratio statistic* to explore the importance of the random walk component within GNP and to compare it to its stationary component.

In this paper, the technique of Cochrane is applied to four time series of the *Federal Republic of Germany* (FRG): the gross domestic product (GDP), the capital stock K, the labour force L and the capital intensity $r = K/L$. The investigations show that the four time series appear to be difference stationary at least of the first order, exhibiting random shocks.

Then, the seminal differential equation of Solow (1956) is revisited. In this context, the capital $K(t)$, the labour force $L(t)$, the capital intensity $r(t)$ are re-considered as piece-wise continuous functions of the time variable t. The emerging non-continuities, which may occur at each discrete period of measurement of $K(t)$ and $r(t)$, are able to represent endured economic shocks, like the oil shock of 1973, and are modelled as jumps. The appropriate mathematical tools to solve these differential equations are the *Impulsive Differential Equations*, (see Bainov and al, 1989-1998). The Solow equation becomes the impulsive Solow equation for piece-wise continuous functions $K(t)$ and $r(t)$.

2. The Solow equation

In this section Solow's neoclassical model, leading to his seminal equation is sketched and a solution exemplified for the strict Cobb-Douglas production function.[1] The simplest and most widely used aggregate production function of neoclassical theory is:

[1] In his famous book on economic growth Maurice FitzGerald Scott (1989, p. 69) evokes that 'orthodox' neoclassical growth theory is concerned with the theories developed by known economists as Johansen, Meade, Phelps, Samuelson, Solow, Swan, Tobin, von Weizäcker, and Yaari, building on earlier ideas coming from Domar, Harrod, Hicks and Robinson. Scott (1989, p. 70) states: "One *conceptual apparatus* of which all make use is the *production function* with total output dependant on inputs of capital and labour, with diminishing returns to each of these factors, and with technical progress shifting the function through time. This is the *centerpiece of neoclassical theory.*"

$$Y = F(K, L) \tag{1}$$

with output Y, capital K, labour L. The variables K, L and Y are also considered independent of time at a first stage.

The Harrod-Domar growth model presents the analysis of long-run growth as a "knife-edge" - any deviation from the path would result in disequilibrium. Robert M. Solow (1956) and Trevor Swan (1956) however contested this conclusion. They claimed that the capital-output ratio K/Y was not constant and that, in fact, it was precisely the adjustment mechanism that would lead a system back to its equilibrium growth path.

Solow (1956, pp. 65-69) considers a full-employment dynamic economy, the main variables being dependent on a continuous time variable t: the aggregate output $Y = Y(t)$, the capital $K = K(t)$, the exponentially growing labour force $L = L(t) = L_0 e^{nt}$, $n > 0$. Solow (1956, p. 66) states that "output is produced with the help of two factors of production, namely capital and labor. Technological possibilities are represented by a production function" (1). Solow then develops his *seminal differential equation* for the rate of increase of the capital stock (investment), the so-called Solow equation

$$\dot{K(t)} = sF(K(t), L_0 e^{nt}), \tag{2}$$

that may be solved for a production function of strict Cobb-Douglas type, $Y = AK^{\alpha}L^{1-\alpha}$, with the technological constant $A = 1$ and the *production elasticity of capital* α, $0 \leq \alpha < 1$, exhibiting a time stationary time path for the capital intensity $r(t) := K(t)/L(t)$. The marginal propensity to save s being a constant, the solution of (2) for the capital intensity is

$$r(t) = [(r(0)^{1-\alpha} - \frac{s}{n})e^{-n(1-\alpha)t} + \frac{s}{n}]^{\frac{1}{1-\alpha}} \xrightarrow[t\to\infty]{} (\frac{s}{n})^{\frac{1}{1-\alpha}}, \tag{3}$$

in accordance with a 'stylised fact' of Kaldor, stating that the *capital-labour ratio* $r(t)$, under the assumption that $r(0)^{1-\alpha} - \frac{s}{n} < 0$, grows at a positive rate, showing no tendency to diminish, (see Valdes, 1999, p. 11).[2]

3. Does the oil shock of 1973 have a permanent impact on GNP?

Solow (1956) has represented the capital K, the labour force L and the capital intensity $r = K/L$ by continuous real functions of time t. Later the nature of macroeconomic time series has been investigated very carefully. Specially the

[2]Kaldor (1961) laid down the so-called 'stylised facts' at a time when there were fewer data accessible than we have today. But nonetheless he was able to identify a number of *empirical regularities* of economic growth which are now regarded as *benchmark*, the minimum requirement any economic model of economic growth must justify. From the 'stylised facts' many others follow, which have been observed in real economies.

role of economic shocks, (see Snowdon, 1994, p. 244), have been investigated. In this section results on this subject are presented.

The conventional approach has been to imagine that the economy evolves along a path, reflecting an underlying trend rate of growth described by Solow's neoclassical model (Solow, 1956). This approach assumes that the long-run trend component of GNP is smooth, with short-run fluctuations about trend being primarily determined by demand shocks. This conventional wisdom was accepted by the Keynesians, monetarists and new classical economists alike until the early 1980s, (see Snowdon, 1994, p. 240-241).

The time series models commonly used to describe temporary deviations are the *trend stationary processes* (TS), where the secular or growth component is modelled by a polynomial in time t, like a linear function $\alpha + \beta t$, and the perturbation z_t, modelled by a stationary and invertible ARMA process $\phi(L)z_t = \theta(L)u_t \ ; \ u_t \sim iid(0, \sigma_u^2)$ with the lag operator L, $L^k y_t = y_{t-k}$, and presented in the form

$$\begin{aligned} y_t &= \alpha + \beta t + z_t \\ \phi(L)z_t &= \theta(L)u_t \quad ; \quad u_t \sim iid(0, \sigma_u^2) \end{aligned} \tag{4}$$

where $\phi(L)$ and $\theta(L)$ are polynomials, satisfying the stationary and invertibility conditions.

Cochrane (1988) formulates the TS models (4), using the property of invertibility of z_t to present them as a moving average process

$$y_t = \alpha + \beta t + \sum_{j=0}^{\infty} a_j \cdot \varepsilon_{t-j} \ , \tag{5}$$

where $\alpha + \beta t$ describes a linear trend and ε_t is a random disturbance, the real coefficients $a_j \xrightarrow[j \to \infty]{} 0$.

A simple form of a TS process (5) for $\beta = 0$ is an $AR(1)$ process with drift α' of the form

$$y_t = \alpha' + \gamma y_{t-1} + \varepsilon_t \quad 0 < \gamma < 1, \tag{6}$$

(see Snowdon, 1994, p. 241, and see Enders, 1995, p. 12, p. 46). But, there is a limited number of large identifiable economic shocks. The understanding of shocks due to significant changes in productivity, like the productivity oil shock of 1973, has become a specific subject of economic thought. It has been asked, if the disturbances due to such shocks die out, meaning that the economy gradually returns to a trend growth path.

Alternatively, in a less conventional approach, it has also been asked, if the impact of the productivity shocks to macroeconomic time series is persistent forever.[3] Nelson and Plosser (1982, p. 142) have treated the question of the appropriate statistical representation of non-stationary macroeconomic time series that have the tendency to move away from an initial state. In this case, the series are represented by a class of non-stationary $ARMA$ processes. The counter-part of the linear TS process is the first-order *difference stationary processes* (DS) (7) of Nelson and Plosser (1982).

$$\begin{aligned} y_t &= \alpha + y_{t-1} + z_t, \\ \phi(L)z_t &= \theta(L)u_t \ ; \ \ u_t \sim iid(0, \sigma_u^2) \end{aligned} \tag{7}$$

where $\phi(L)$ and $\theta(L)$ are polynomials satisfying the stationary and invertibility conditions.

The simplest member of that class (7) is the *random walk* process for which the changes z_t are serially uncorrelated, represented in the following form

$$y_t = \alpha + y_{t-1} + z_t. \tag{8}$$

The difference between the TS models and DS models is dramatic. Since y_t depends on y_{t-1} the shock is transmitted in time. In the conventional TS approach, modelled by (4), (5), the impact of the shock generally dies out, since in the DS approach, modelled by (7), (8) the effect is *permanent.*

Consequently, if Nelson and Plosser (1982) are right, the impact of the oil shock of 1973 has a permanent effect on output.

4. How big is the random walk in macroeconomic time series?

The result of Nelson and Plosser (1982) has been questioned. Thus, Cochrane (1988) re-examined the long-run properties of GNP and argued that GNP does in fact *revert toward a 'trend'* following a shock and proposes models for macroeconomic time series that can partly consist of a random walk plus a trend stationary component. It is investigated, how much long-run forecasts in this case do respond to supply-side or demand shocks.

4.1. Unit roots and the variance ratio statistic

When the macroeconomic time series are described by DS models, there is the necessity to investigate their degree of integration. That means, the presence of unit roots has to be evaluated. The Dickey-Fuller (DF), the augmented Dickey-Fuller (ADF) and the Phillips-Perron (PP) tests are frequently used

[3] As an example, the *real business cycle theory* emphasises the importance of real productivity shocks over monetary shocks in the generation of economic fluctuations, (see Turner, 1993, p. 10).

unit root tests, based on the Box-Jenkins methodology. Nelson and Plosser (1982) applied the DF tests to many US time series and showed that macroeconomic variables are well described by ARIMA models with one unit root, (see Maddala and In-Moo-Kim, 1998, p. 47).

But, it is known that there are several problems with the DF, ADF and PP tests. They specially have a low power, (see Maddala and In-Moo-Kim, 1998, p. 98). For this reason, new methods to investigate the presence of unit roots in macroeconomic time series have been developed. Cochrane (1988) proposes for the *measurement of persistence* to evaluate the presence of a unit root with the *variance ratio (VR) statistic* (VR_p).

The VR_p is the *variance of the p-differences* of a time series $\{y_t\}$ divided by p times the *variance of the* 1-*differences* of the time series $\{y_t\}$. Let $\sigma_p^2 = p^{-1}var(y_t - y_{t-p})$ be $1/p$ times the *variance of the p-differences* and $\sigma_{\Delta y}^2 = var(y_t - y_{t-1})$ be the *variance of the* $1-$*differences*. Then, the variance ratio statistic is $VR_p = \frac{\sigma_p^2}{\sigma_{\Delta y}^2}$.

Cochrane uses the following estimator for the variance ratio

$$\widehat{VR}_p = \frac{var(y_t - y_{t-p})}{p \cdot var(y_t - y_{t-1})} \cdot (\frac{T}{T-p+1}), \tag{9}$$

defined for each lag p. Consider a series $\{y_t\}$ of T observations, then the *variance ratio statistic* VR_p with a bias ratio corrector $\frac{T}{T-p+1}$ for short time series, measures the variance of the p-differences of the series divided by p to the variance of the 1-differences of the series $\{y_t\}$. It is well-known that if $\{y_t\}$ is a pure random walk (8), then $VR_p \rightarrow 1$ for $p \rightarrow \infty$. If $\{y_t\}$ is trend-stationary, then $VR_p \rightarrow 0$ for $p \rightarrow \infty$.

Beveridge and Nelson (1981) have shown that every 1-difference stationary process $I(1)$ can be decomposed into a *random walk component* and a *stationary component*. Under the assumption that $\{y_t\}$ is a 1-difference stationary process, this decomposition is applied to $\{y_t\}$ and gives

$$y_t = x_t + c_t, \tag{10}$$

where $\{x_t\}$ is the random walk (8) component and $\{c_t\}$ is the stationary component. The innovation variance of the random walk component $\sigma_{\Delta x}^2 = var(x_t - x_{t-1})$ is a natural measure of the importance of the random walk component and is put into relation with the variance $\sigma_{\Delta y}^2 = var(y_t - y_{t-1})$. Cochrane (1988, p. 906) shows that the limit of σ_p^2 is the innovation variance of the random walk component,

$$\lim_{p\rightarrow\infty} \sigma_p^2 = \lim_{p\rightarrow\infty} VR_p \cdot \sigma_{\Delta y}^2 = \sigma_{\Delta x}^2. \tag{11}$$

4.2. Cochrane's empirical results with the GNP of the USA

Cochrane (1988, p. 989) developed the idea that the fluctuations in GNP are partly permanent and partly temporary, which one can model as a combination of a stationary series and a random walk. If the series $\{y_t\}$ represents the $\log(GNP)$ of the USA, then the plot of $var(y_t - y_{t-p})/p = VR_p \cdot var(y_t - y_{t-1})$ versus p settles down to the variance of the shock to the random walk component $\{x_t\}$ (10) of the series $\{y_t\}$. If the series $\{y_t\}$ represents the log real per capita GNP of the USA, $1/p$ times the variance $var(y_t - y_{t-p})$ of the p-differences settles down to about *one-third* of the variance of the 1-differences $var(y_t - y_{t-1})$. Cochrane suggests that the innovation variance of the random walk component $\sigma^2_{\Delta x} = var(x_t - x_{t-1})$ is about one-third of the variance of year-to-year changes: annual growth rates of GNP contain a large temporary component. Cochrane concludes that "if there is a *random walk component* in GNP at all, it is small", (see Cochrane, 1988, p. 915).

5. Empirical results with German macroeconomic time series

In this section the results of empirical investigations concerning the presence of unit roots or of a random walk component within the four German macroeconomic time series are described.

5.1. The choice of German macroeconomic time series

The annual German macroeconomic time series of $GDP(t)$, capital stock $K(t)$, capital intensity $r(t) = K(t)/L(t)$, all at prices of 1991 and the labour force $L(t)$, available within the period of $1950 - 2000$ have been chosen to perform that empirical analysis.[4] All the values of the four series are expressed at constant prices of the year 1991. Usually, an exponential growth is assumed to be present in these four macroeconomic time series.[5]

The sixteen pictures of Fig. 1 present in the four columns the German time series, the $1-$differences of the series, the logged series and the $1-$differences

[4]The 'Statistisches Bundesamt' (2000, Volkswirtschaftliche Gesamtrechnung) publishes time series of the former and the reunified *Federal Republic of Germany* (FRG). Consequently, in order to get series of a maximal number of observed values, the time series $1991 - 2000$ of the enlarged FRG have been linked to the time series of the former FRG for $1950 - 1991$. The year 1991 is the breaking period between both time series. For example: In the former FRG, the GDP of the year 1991 was $GDP1_{1991} = 2713$, in the reunified FRG, the GDP is $GDP2_{1991} = 2938$. Thus, all the values $GDP2_t; t = 1991, 1992, ..., 2000$ have to be multiplied by the scaling factor $c = \frac{2713}{2938} = 0.9234$, in order to link both GDP series.

[5]The discrete rate of growth, (see Turner, 1999, p. 7 or, see Cochrane, 1998, p. 894), of GDP is given by $\Delta \log(GDP) \sim \frac{GDP_t - GDP_{t-1}}{GDP_{t-1}}$ and is considered as a measure of the *rate of inflation.*

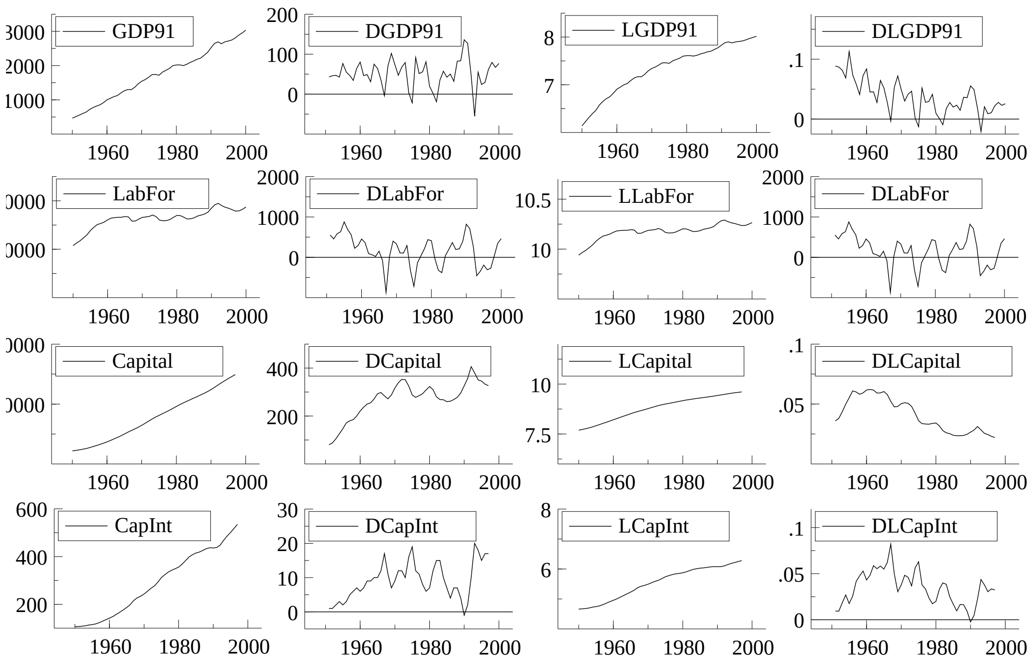

Figure 1: German Macroeconomic Time Series (PcGive)

of the logged series. In the four rows are the GDP, the labour force L, the capital K and the capital intensity r.

5.2. The presence of random walk in the German series

In a first approach, the presence of unit roots has been investigated in the four initial and logged German series, using the ADF(p)-test for the lags $p = 0, 1, 2$ without constant.[6] Clearly, from the ADF tests none of the German time series GDP, L, K, r exhibits stationarity, the four series appear to be at least $I(1)$. But the low power of these tests must be taken into account. - Thus, the method of Cochrane (1988) with the variance ratio statistic (9) has been applied to the present German series. Variance ratio values have been calculated for each of the sixteen series of Fig. 1. The results are presented

[6]The results are obtained with PcGive 9.00: The computational results are illustrated for $p = 0$. Thus, the DF test results are given for $T = 50$ observations of the four series GDP, $L(t)$, $K(t)$, $r(t)$ and its logged series. The critical values of significance level $\alpha = 5\%$ remain identical and are noted as $c(\alpha = 5\%) = -1.948$. The results suggest the presence of unit roots. For GDP: DF $= 8.210 > -1.948$; for L: DF $= 2.440 > -1.948$; for the stock of capital K: DF $= 20.098 > -1.948$; for r: DF $= 11.320 > -1.948$. For the logged series the results are analogous. For $log(GDP)$: DF $= 8.243 > -1.948$; for $log(L)$: DF $= 2.724 > -1.948$; for $log(K)$: DF $= 16.086 > -1.948$; for $log(r)$: DF $= 12.095 > -1.948$.

with the sixteen plots of Fig. 2. Obviously none of them shows convergence to 0. This means, none of them shows pure stationarity.

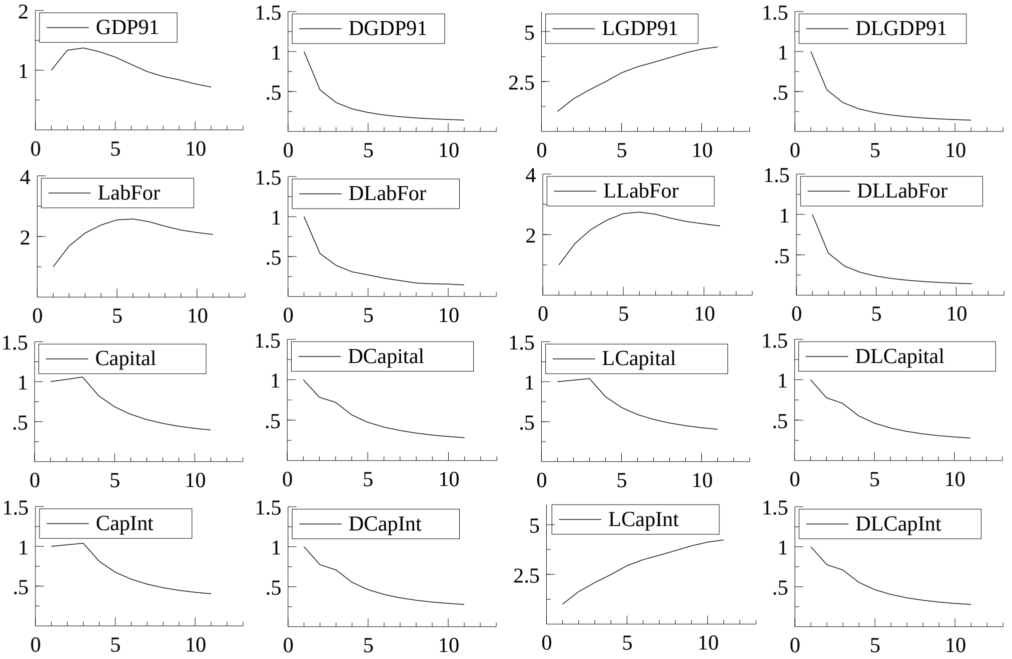

Figure 2: Variance ratio statistics of the German Macroeconomic Time Series, eleven lags (PcGive)

The persistence of the random walk component is explored with the variance ratio statistic (9). The computational results (Tab.1)[7] support the hypothesis of the presence of a *random walk component* and therefore also a unit root in each of the German series GDP, L, K, r and $log(GDP)$, $log(L)$. Therefore, the German series GDP, L, K, r and the series $log(GDP)$, $log(L)$ appear to be at least integrated $I(1)$.

Next, the integration of the capital intensity $r(t)$ is considered. The ADF(p) test for lag $p = 1$ applied to the $1-$differenced time series $\Delta r(t)$ show that the null hypothesis of a unit root is rejected.[8] Then, the integration of the capital

[7]Cochrane (1988, p. 894-895) exemplifies that the variance ratio test values $VR_p > 1$ reveal "series that will continue to diverge from its previously forecast value following a shock. Campbell and Mankiw (1987) originated and emphasise this interpretation."

[8]The ADF(p) tests for lag $p = 0, 1, 2, 3, 4$ with constant or with constant and trend are applied to $r(t)$ and exhibit no stationarity. The results of the ADF tests for lag $p = 1$ *with contant* or *with constant and trend* for the series $\Delta r(t)$ at the significance level $\alpha = 5\%$ are shown. The critical value is $c(\alpha = 5\%)$.
ADF(1) with constant; $ADF(DCapint) = -3.356^* < -2.927 = c(\alpha = 5\%)$; ADF(1) with constant and trend; $ADF(DCapint) = -3.693^* < -3.511 = c(\alpha = 5\%)$. On the other side,

Lags p	*GDP*	*L*	*K*	*r*	*log(GDP)*	*log(L)*	*diff(K)*	*diff(r)*
0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	1.33	1.69	1.02	1.02	1.63	1.71	0.78	0.78
2	1.37	2.11	1.06	1.02	2.09	2.17	0.72	0.71
3	1.31	2.38	0.82	0.81	2.49	2.48	0.56	0.56
4	1.22	2.55	0.68	0.68	2.94	2.69	0.47	0.46
5	1.10	2.58	0.59	0.59	3.24	2.74	0.41	0.40
6	0.98	2.49	0.53	0.52	3.46	2.67	0.37	0.36
7	0.90	2.33	0.48	0.48	3.69	2.53	0.34	0.33
8	0.84	2.21	0.44	0.45	3.93	2.42	0.32	0.31
9	0.77	2.13	0.42	0.42	4.12	2.35	0.30	0.29
10	0.72	2.06	0.40	0.40	4.23	2.29	0.28	0.28

Table 1: Variance Ratio of four German macro time series

$K(t)$ is considered. The ADF-test results show that the null hypothesis of a unit root is not rejected in the time series $K(t)$ and $\Delta K(t)$ in accordance with the variance ratio statistic of $\Delta K(t)$ (Tab.1).[9]

5.3. Jumps in the functions of German capital and capital intensity

The present empirical results permit to support the hypothesis that the German annual time series of *capital intensity* $r = K/L$ is modelled as an I(1) time series, the Beveridge (1981) decomposition scheme becoming applicable, $r(t)$ is decomposed into a *random walk* component and a *stationary* component (10). The German annual time series of *capital* $K(t)$ appears to be I(2).

The main proposition of this paper is to model the jumps at each of the annual periods of measurement of the series $r(t)$ and $K(t)$ and to revisit the Solow equation (2) with real-valued *piece-wise continuous functions.*

6. Impulsive differential equations

The presently developed ideas has led the authors to apply the sophisticated mathematical tools of *Impulsive differential equations (IDE)* to solve the seminal differential equation of Solow (1956) (2) for piece-wise continuous functions with jumps, representing either the I(1) capital intensity $r(t)$, respectively the I(2) capital $K(t)$, making that Solow equation *impulsive.*

the variance ratio statistic of $\Delta r(t)$ (Tab.l) does not converge rapidly to zero.

[9]The ADF tests for lags $p = 0, 1, 2, 3, 4$ *without constant*, *with contant* or *with constant and trend* for the series $K(t)$ and $\Delta K(t)$ at the significance level $\alpha = 5\%$ have been performed and showed that the presence of unit roots could not be refused in none of the cases, whereas for $\Delta^2 K(t)$ the null hypothesis of unit roots can be excluded on the basis of ADF tests.

6.1. Preliminaries

The impulsive differential equations can be successfully used for mathematical simulation of processes and phenomena which are subject to short-term perturbations during their evolution. The duration of the perturbations is negligible in comparison with the duration of the process considered, and they can be thought of as momentary. The theory of impulsive ordinary differential equations started with the pioneer paper of V. Mil'man and A. Myshkis (1960) and has been an object of intensive investigations during the last three decades, (see Bainov, 1989-1998, and see Samoilenko, 1987). Such a generalization of the notion of a differential equation provides a natural framework for mathematical simulation of many processes and phenomena in theory of optimal control, biology, population dynamics, bio-technologies, medicine, electronics, radio engineering and economics.

6.2. Statement of the problem and preliminary notes

Let R^n be the n-dimensional Euclidean space with elements $x = col(x_1, ..., x_n)$ and norm $|x| = (\sum_{k=1}^{n} x_k^2)^{\frac{1}{2}}$; $R_+ = [0, \infty)$; $t_0 \in R_+$; $x_0 \in R^n$. Let $\tau_0 < \tau_1 < \tau_2 < ... < \tau_k < ....$

Denote by $x(t) = x(t; t_0, x_0)$ the solution of the initial value problem

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)),\ t > t_0,\ t \neq \tau_k, \\ x(t_0 + 0) &= x_0, \\ \Delta x(t) &= I_k(x(t)),\ t = \tau_k,\ k = 1, 2, ..., \end{aligned} \tag{12}$$

where $f : (t_0, \infty) \times R^n \to R^n$, $I_k :\ R^n \to R^n$, $\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0),\ k = 1, 2, ...$

For $t \in (\tau_k, \tau_{k+1}]$ the solution $x(t)$ of the problem (12) satisfies the system $\frac{dx}{dt} = f(t, x(t))$ and for $t = \tau_k$ - the relations $x(\tau_k + 0) = x(\tau_k) + \Delta x(\tau_k) = x(\tau_k) + I_k(x(\tau_k))$.

6.3. IDE modelling of the Solow equation

Let us consider the Solow equation (2). According to the method of Cochrane (1988) and considering the present empirical results in the German time series it is clear that during the process of the growth, the capital $K = K(t)$ can be subject to short-term perturbations at certain moments of time. We note that the change by jumps of the capital $K = K(t)$ and of the capital intensity $r(t) = \frac{K(t)}{L(t)}$ is intrinsic to the system itself.

Let us note by $\tau_1, \tau_2, ..., \tau_k > 0$ the moments, when the total stock of capital $K(t)$ is subject to shock effects due to which it changes from the positions

$K(\tau_k - 0)$ into the position $K(\tau_k + 0)$. Then it is not reasonable to expect a regular solution of the equation (2). Instead, the solution must have some jumps at these moments and the jumps follow a specific pattern. An adequate mathematical model of the growth of capital in this case will be the impulsive differential equation of the form:

$$\begin{aligned} \dot{K}(t) &= sF(K(t), L_0 e^{nt}),\ t > t_0,\ t \neq \tau_k, \\ K(t_0 + 0) &= K_0 \\ \Delta K(t) &= J_k(K(t)),\ t = \tau_k,\ k = 1, 2, ..., \end{aligned} \tag{13}$$

where functions J_k characterize the magnitude of the impulse effect at the times τ_k; $K(\tau_k - 0)$ and $K(\tau_k + 0)$ are respectively the capital level before and after the impulsive effect; K_0 is the initial capital.

The initial value problem (13) is a generalization of the Solow growth equation of capital and of the result of Cochrane (1988). It can be used in the economic studies of business cycles in situation when the total stock of capital $K(t)$ is subject to shock effects. Particularly, the case $\Delta K(\tau_k) < 0$ corresponds to instantaneous reduction of the capital at times $\tau_k,\ k = 1, 2, ...$, while the case $\Delta K(\tau_k) > 0$ describes heavy intensification of the capital.

By means of the models of type (13) it is possible to investigate one of the most important problems of economics - the problem of the optimal control of the business cycles, (see May, 1973).

In the model (13) the moments of impulse effect is caused by an inferior effect. But the moments of impulse effect can be caused by an exterior effect.

1) The moments of jumps $\tau_1, \tau_2, ..., \tau_k > 0$ may be a given/fixed/ sequence of moments of impulse effect;

2) The moments of impulse effect may come when some spatial-temporal relation $\Phi(t, K) = 0$ is satisfied, i.e. when the mapping point of the process meets the surface of equation $\Phi(t, K) = 0$. The equation with this kind of impulse effect is said to be the equation with variable impulsive perturbations.

7. Summary and conclusion

A straightforward calculation shows that the solution $x(t)$ of problem (12) satisfies the equation:

$$x(t) = \begin{cases} x_0 + \sum\limits_{t_0 \le \tau_k < t} I_k(x(\tau_k)) + \int_{t_0}^{t} f(s, x(s))ds,\ t \in J^+, \\ x_0 - \sum\limits_{t \le \tau_k \le t_0} I_k(x(\tau_k)) + \int_{t_0}^{t} f(s, x(s))ds,\ t \in J^-. \end{cases} \tag{14}$$

where $J^+ = J^+(t_0, x_0)$ is the maximal interval of type $[t_0, \beta)$ where the solution $x(t) = x(t; t_0, x_0)$ is defined and $J^- = J^-(t_0, x_0) = (\alpha, t_0)$.

Therefore, the solution of the impulsive Solow equation (13) for the capital with the production function of Cobb-Douglas satisfies the equation

$$K(t) = [K(t_0)^{1-\alpha} - \frac{s}{n} L_0^{1-\alpha} + \frac{s}{n} L_0^{1-\alpha} e^{n(1-\alpha)t}]^{\frac{1}{1-\alpha}} + \sum_{t_0 \leq \tau_k < t} J_k(K(\tau_k)), \quad (15)$$

i.e. it consists of a continuous and a discrete component, in accordance with the present investigations of the four German series. The discrete component is of a great importance for the study and examination of characteristics and regulation of the $K(t)$. For the capital intensity $r(t) = \frac{K(t)}{L(t)}$ we have

$$r(t) = [(r(t_0)^{1-\alpha} - \frac{s}{n})e^{-n(1-\alpha)t} + \frac{s}{n}]^{\frac{1}{1-\alpha}} + \sum_{t_0 \leq \tau_k < t} B_k(r(\tau_k), \quad (16)$$

where $B_k(r(\tau_k)) = r(\tau_k + 0) - r(\tau_k),\ \ k = 1, 2, ...$

It is easily seen that if $L(\tau_k+0) = L(\tau_k)$ then $B_k(r(\tau_k)) = [L_0 e^{nt}]^{-1} J_k(K(\tau_k))$ and $r(t) \to (\frac{s}{n})^{\frac{1}{1-\alpha}}$ as $t \to \infty$. But if $L(\tau_k + 0) \neq L(\tau_k)$ for some $k = 1, 2, ...$, then the equilibrium value of the capital-labour ratio depends on the shocks at the moments τ_k, $k = 1, 2, ...$

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