Partially Observed Stochastic Epidemics

Victor M. Panaretos

Institut de Mathématiques

Ecole Polytechnique Fédérale de Lausanne

victor.panaretos@epfl.ch
http://smat.epfl.ch

Stochastic Epidemics

Stochastic Epidemic = *stochastic model for evolution of epidemic*

Often$\color{red}{\rightarrow}$ $\begin{cases} \text{stochastic model} = \text{stochastic process } (T = \mathbb{R}_+, \mathbb{Z}_+) \\ \text{evolution} = \text{“counting”} \end{cases}$

Purposes:

- Description (behaviour of epidemic)
- Prediction (outcome of epidemic)

long history + rich literature

Origins + "Recent" Themes

1949: Bailey, $\sim$ 1950+: Bailey, Whittle, Bartlett, Kendall
$\hookrightarrow$ stochastic variants of Kermack-McKendrick deterministic model

followed by explosion of literature...

"Recent" Themes:

- Model approximation
  (Ball & Barbour [1990, JAP], Ball & Donnelly [1995, Stoc. Proc. App.])
- Threshold theorems
  (Andersson & Djehiche [1998, JAP], von Bahr & Martin-Löf [1980, AAP])
- Time to extinction
  (Barbour [1975, Biometrika], Nåsel [1999, JRSSB])
- Random graph interpretations
  (Barbour & Mollison [1990, Springer Lec. Notes])

$\hookrightarrow$ under variety of different assumptions (= models)

Outbreak of an Epidemic

Offset of an epidemic: emergency *control rules* justified?

Control rules:

- may decisively affect epidemic
- have grave cost

Must decide on basis of initial observations:

- Typically look at $\mathbb{P}[\mathcal{E}_0]/\mathbb{P}[\mathcal{E}_\infty]$

Depends on what $\mathbb{P}$ is! (model)
$\hookrightarrow$ and interpretation of observations...

Offset in Large Populations

Stochastic models of initial stages of epidemic?

In practice have finite population:

$$N = \underset{\textit{susceptibles}}{S} + \underset{\textit{infectious}}{I} + \underset{\textit{removed}}{R}$$

Assuming large population of susceptibles $\longrightarrow$ Approximation by simple stochastic models:

- Birth and Death Processes (continuous time)
- Branching Processes (discrete time)

Extinction/explosion duality: $\mathbb{P}[\mathcal{E}_0] + \mathbb{P}[\mathcal{E}_\infty] = 1$

Both manifest phase transitions:

- Birth/Death rate ratio $\rho \substack{<\\=\\>} 1$
- Offspring distribution mean $\mu \substack{<\\=\\>} 1$

Incorporating Data Information

Criticality results $\longrightarrow$ depend on general specifications

Notation:

$$Z_t = \#(\text{infectious individuals at time } t)$$

Include history (data) up to present, t_0, via conditioning:

$$\mathbb{P}[\mathcal{E}_0 | \{Z_t\}_{0 \leq t \leq t_0}] = \mathbb{P}[\mathcal{E}_0 | Z_{t_0}] \tag{1}$$

Markov property: current value is predictively sufficient

QUESTION: Do we really completely observe $\{Z_t\}$?

Partially Observed Stochastic Epidemics?

D.G.Kendall [1956, Proc. 3rd Berkeley Symp.]:

- In practice do not observe infections
- In deterministic models observe a proportion, but stochastic case more involved

D.R. Cox [recently, personal communication]:

- "Solvable" perturbation of classical models to take partial observation into account?

An approach: V.M. Panaretos [2007, J. Math. Biol.]

General Considerations

In practice:

- cases recorded in discrete time (e.g. every day)
- some cases undetected
- observed cases are quarantined

Suggests:

- $T = \mathbb{Z}_+$
- observations $Y_n = f(Z_n, \xi_n), \quad \xi_n \perp Z_n$
- interventions $Z_{n+1}|Z_n \overset{d}{\neq} Z_{n+1}|(Z_n, Y_n)$

Want:

- tractability
- interpretable marginal model

A Simple Emission/Intervention Mechanism

- Emissions:
  Each one of Z_n independently overlooked with probability θ
  $\hookrightarrow Y_n|Z_n \sim \text{Binom}(1-\theta, Z_n)$

- Interaction:
  Observed individuals cannot spread disease. Each unobserved individual independently produces identically distributed "offsprings"
  $\hookrightarrow Z_{n+1} \stackrel{d}{=} \sum_{k=1}^{Z_n - Y_n} \xi_{k,n}, \quad \{\xi_{k,n}\}$ iid

Simple Coupling Construction

Define a stochastic process $\{(Z_n, Y_n)\}_{n\geq 0}$ as follows:

$$\{\xi_{i,n}\} \text{ iid},\ \ B \sim \text{Bernoulli}(1-\theta) \begin{cases} Z_0 = N_0, & \\ Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{i,n}(1 - B_{i,n}) & n \geq 1 \\ Y_n := \sum_{i=1}^{Z_n} B_{in}, & n \geq 0 \end{cases} \tag{2}$$

Branching process framework:

- framework includes discretely sampled birth/death processes as special case

Conditioning on Y_n Instead of Z_n

Joint process $\{(Z_n, Y_n)\}$ a Markov chain on $\mathbb{Z}_+^2$

Can obtain explicit k-step transition pgfs

$$P_n(r,s) = \left\{ \frac{1}{\theta}\gamma_n(r\theta + rs(1-\theta)) - \frac{1-\theta}{\theta} \right\}^{z_0 - y_0}$$

How do things differ when conditioning on the observable component Y_n?

E.G. two issues of basic interest:

- Conditional probability of extinction
- Distribution of conditional time to extinction

Will need k-step prediction distribution (actually generating function):

$$\mathbb{P}[Z_{n+k} = \cdot | Y_n = y_n]$$

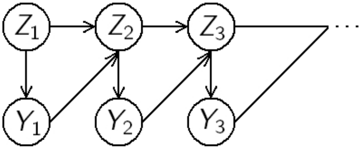
Conditioning on the First Step

Basic property $\longrightarrow$ $\{Z_n\}$ marginally a BRANCHING PROCESS

Use "self-similarity" property for prediction distribution

$\{Z_n\}$-type branching process started at dist$\{Z_n|Y_n\}$?

Need to be careful at first step:



Getting there...

Following sequence of steps:

- Getting from $Z_n|Y_n$ to Z_{n+1}
  $\hookrightarrow$ Requires $G_{Z_n|Y_n}$
- Hence obtain $G_{Z_{n+1}|Y_n}$
- Use "self-similarity"

$$\left.\begin{array}{l}\text{Marginal Branching}\\ \text{Binomial Thinning}\\ \text{Conditional Independence}\end{array}\right\} \longrightarrow \text{Crucial in all steps.}$$

Exact Expressions

- Probability of Extinction:

$$\begin{aligned}
\mathbb{P}_\mu[\mathcal{E}_0 | Y_n = y] &= \sum_{z=0}^{\infty} \mathbb{P}_\mu[\mathcal{E}_0 | Z_{n+1} = z, Y_n = y] \mathbb{P}_\mu[Z_{n+1} | Y_n = y] \\
&= \sum_{z=0}^{\infty} (s_0)^z \mathbb{P}_\mu[Z_{n+1} | Y_n = y] \\
&= G_{Z_{n+1} | Y_n = y} \\
&= \frac{\lambda_n^{(y)}(\theta G_\xi(s_0))}{\lambda_n^{(y)}(\theta)}
\end{aligned}$$

- Time to Extinction

$$\begin{aligned}
\mathbb{P}_\mu[T_0 \leq n + k | Y_n = y] &= \mathbb{P}_\mu[Z_{n+k} = 0 | Y_n = y] \\
&= G_{Z_{n+k} | Y_n = y}(0) \\
&= \frac{\lambda_n^{(y)}(\theta G_\xi(\gamma_{k-1}(0)))}{\lambda_n^{(y)}(\theta)}
\end{aligned}$$

Conditioning Further into the Past via "Bootstrapping"

Saw what happened when replacing:

- Z_n the true current value

by

- Y_n what we really observe

BUT: Y_n is predictively insufficient (Markov "breaks down").
Should condition on whole history $Y_1, \ldots, Y_n$
Can show:

$$\text{(A)} \quad G_{Z_{n+1}|(Y_n=y_n,\ldots,Y_1=y_1)}(s) = \frac{G_{Z_n|(Y_n,\ldots,Y_1)}(G_\xi(s))}{(G_\xi(s))^{y_n}}$$

$$\text{(B)} \quad G_{Z_n|(Y_n=y_n,\ldots,Y_1=y_1)}(s) = \frac{G^{(y_n)}_{Z_n|(Y_{n-1},\ldots,Y_1)}(s\theta)}{G^{(y_n)}_{Z_n|(Y_{n-1},\ldots,Y_1)}(\theta)}$$

Start from time origin, bootstrap upwards by iteratively switching:

$$G_{Z_2|Y_1} \overset{(B)}{\longrightarrow} G_{Z_1|(Y_2,Y_1)} \overset{(A)}{\longrightarrow} G_{Z_3|(Y_2,Y_1)} \overset{(B)}{\longrightarrow} G_{Z_3|(Y_3,Y_2,Y_1)} \overset{(A)}{\longrightarrow} G_{Z_4|(Y_3,Y_2,Y_1)} \overset{(B)}{\longrightarrow} \ldots$$

A More General Model

$$\xi, \zeta \text{ iid}, \; B \sim \text{Bernoulli}(1-\theta) \begin{cases} Z_0 = N_0, \\ Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{i,n}(1 - B_{i,n}) + \sum_{i=1}^{Z_n} \zeta_{i,n} B_{i,n}, \\ Y_n := \sum_{i=1}^{Z_n} B_{in}, \end{cases} \tag{3}$$

Analysis via conditional independence (simple convolution with "simpler model"):

$$\sum_{i=1}^{Z_n} \xi_{i,n}(1 - B_{i,n}) \Big| Y_n \qquad \coprod \qquad \sum_{i=1}^{Z_n} \zeta_{i,n} B_{i,n} \Big| Y_n \tag{4}$$

Some Remarks

- Small perturbation increases complication
  - Inherent non-linearities make generalisations intractable
  - Took advantage of "solvable" marginal model given emission and interaction mechanism
- Conditioning on the whole past $\sigma(Y_1, Y_2, ..., Y_n)$
- Loss of information
- Inference given observable data (pseudo-likelihood?) - Crucial Elements:
  - Marginal Model
  - Observation Mechanism ("potential function")
  - Feynman-Kac Path measures?
- Geometric case completely tractable (discretely observed birth-death process...)

A Couple of References

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